October 2021 update

The preparation of the new 20-th edition of the Kourovka Notebook is now underway. New problems can be sent to the editors: Evgeny Khukhro `khukhro@yahoo.co.uk` and Victor Mazurov `mazurov@math.nsc.ru`. We would also appreciate any information about the problems in the previous issues.

See below the latest updates for the current 19th issue, which are all incorporated in the PDF file of the Kourovka Notebook available at `https://kourovka-notebook.org` (see also `http://math.nsc.ru/~alglog/19tkt.pdf` and `https://arxiv.org/pdf/1401.0300.pdf`).

For convenience of the readers all the updates since the first appearance of the 19th edition are also listed below; the problems with the newest comments or solutions added in October 2021 are marked **NEW**.

***3.51.** Is it true that every finite group with a group of automorphisms Φ which acts regularly on the set of conjugacy classes of G (that is, leaves only the identity class fixed) is soluble? The answer is known to be affirmative in the case where Φ is a cyclic group generated by a regular automorphism. *A. I. Saksonov*

*No, not always (Y. Fine, *J. Group Theory*, **22**, no. 6 (2019), 1077–1087).

***6.18.** (Well-known problem). Suppose that a class $\mathfrak{K}$ of 2-generator groups generates the variety of all groups. Is a non-cyclic free group residually in $\mathfrak{K}$? *V. M. Levchuk*

*Not always (S. J. Pride, *Math. Z.*, **131** (1973), 245–248).

NEW ***7.17.** Is the number of maximal subgroups of the finite group G at most $|G|-1$?

Editors' remarks: This is proved for G soluble (G. E. Wall, *J. Austral. Math. Soc.*, **2** (1961–62), 35–59) and for symmetric groups $\mathbb{S}_n$ for sufficiently large n (M. Liebeck, A. Shalev, *J. Combin. Theory, Ser. A*, **75** (1996), 341–352); it is also proved (M. W. Liebeck, L. Pyber, A. Shalev, *J. Algebra*, **317** (2007), 184–197) that any finite group G has at most $2C|G|^{3/2}$ maximal subgroups, where C is an absolute constant. *R. Griess*

*No, not always (R. Guralnick, F. Lübeck, L. Scott, T. Sprowl; see `arxiv.org/pdf/1303.2752`). Infinitely many counterexamples where found in (F. Lübeck, *Trans. Amer. Math. Soc.*, **373**, no. 4 (2020), 2331–2347).

***7.36.** Is it true that every residually finite group in which every subgroup of finite index (including the group itself) is defined by a single defining relation is either free or isomorphic to the fundamental group of a compact surface? *O. V. Mel'nikov*

*No; for example, let $H_n = \langle x, y \mid y^{-1}xy = x^n\rangle$, $n = 2, 3, \dots$; then every subgroup of finite index in H_n is isomorphic to a group H_m for some m (V. A. Churkin, *Abstracts of 8th All-USSR Symp. on Group Theory*, Kiev, 1982, 139–140 (Russian)).

***8.31.** Describe the finite groups in which every proper subgroup has a complement in some larger subgroup. Among these groups are, for example, $PSL_2(7)$ and all Sylow subgroups of symmetric groups. *V. M. Levchuk*

*These groups are described independently in (V. M. Levchuk, A. G. Likharev, *Siberian Math. J.*, **47**, no. 4 (2006), 659–668) and in (V. N. Tyutyanov, *Proc. Gomel' State Univ.*, **3** (2006), 178–183 (in Russian)).

NEW *__9.32.__ What locally compact groups satisfy the following condition: the product of any two closed subgroups is also a closed subgroup? Abelian groups with this property were described in (Yu. N. Mukhin, *Math. Notes*, **8** (1970), 755–760).

*Such groups are described (W. Herfort, K. H. Hofmann, F. G. Russo, *Adv. Math.*, **390** (2021), 107894).

Yu. N. Mukhin

NEW *__9.49.__ Let G be a compact group of weight $> \omega_2$. Is it true that the space of all closed subgroups of G with respect to E-topology is non-dyadic?

I. V. Protasov, Yu. V. Tsybenko

*Yes, it is true (Yu. V. Tsybenko, *Ukrain. Math. J.*, **38** (1986), 542–545).

NEW *__11.2.__ Classify the simple groups that are isomorphic to the multiplicative groups of finite rings, in particular, of the group rings of finite groups over finite fields and over $\mathbb{Z}/n\mathbb{Z}$, $n \in \mathbb{Z}$.

R. Zh. Aleev

*Such a classification is obtained in (C. Davis, T. Occhipinti, *J. Pure Appl. Algebra*, **218**, no. 4 (2014), 743–744).

NEW *__12.22.__ b) Let $\Delta(G)$ be the augmentation ideal of the integer group ring of an arbitrary group G. Then $D_n(G) = G \cap (1 + \Delta^n(G))$ contains the nth lower central subgroup $\gamma_n(G)$ of G. Is it true that $D_n(G)/\gamma_n(G)$ has exponent dividing 2?

N. D. Gupta, Yu. V. Kuz'min

*No, not always (L. Bartholdi, R. Mikhailov, `https://arxiv.org/abs/1805.10894`).

*__12.32.__ Prove an analogue of Higman's theorem for the Burnside variety $\mathfrak{B}_n$ of groups of odd exponent $n \gg 1$, that is, prove that every recursively presented group of exponent n can be embedded in a finitely presented (in $\mathfrak{B}_n$) group of exponent n.

S. V. Ivanov

*It is proved (A. Yu. Olshanskii, *J. Algebra*, **560** (2020), 960–1052).

*__12.38.__ (J. G. Thompson). For a finite group G, we denote by $N(G)$ the set of all orders of the conjugacy classes of G. Is it true that if G is a finite non-abelian simple group, H a finite group with trivial centre and $N(G) = N(H)$, then G and H are isomorphic?

A. S. Kondratiev, W. J. Shi

*Yes, it is true. The final step of the proof is in the paper (I. B. Gorshkov, *Commun. Algebra*, **47**, no. 12 (2019), 5192–5206), which contains references to the previous steps by M. Ahanjideh, N. Ahanjideh, S. H. Alavi, G. Y. Chen, A. Daneshkhah, M. R. Darafsheh, I. B. Gorshkov, A. Iranmanesh, I. Kaygorodov, Behn. Khosravi, Behr. Khosravi, A. Kukharev, W. Shi, A. Shlepkin, A. V. Vasil'ev, L. Wang, M. Xu.

*__12.79.__ Suppose that a and b are two elements of a finite group G such that the function

$$\varphi(g) = 1^G(g) - 1^G_{\langle a\rangle}(g) - 1^G_{\langle b\rangle}(g) - 1^G_{\langle ab\rangle}(g) + 2$$

is a character of G. Is it true that $G = \langle a, b\rangle$? The converse statement is true.

S. P. Strunkov

*No, not always: a counterexample is given by $G = A_4$, $a = b = (123)$. (S. V. Skresanov, *Algebra Logic*, **58**, no. 3 (2019), 249–253).

***12.80.** b) (K. W. Roggenkamp). Is it true that the number of p-blocks of defect 0 of a finite simple group G is equal to the number of the conjugacy classes of elements $g \in G$ such that the number of solutions of the equation $[x, y] = g$ in G is not divisible by p?

S. P. Strunkov

*No, not always. For example in the alternating group A_5 for $p = 2$, the number of 2-blocks of defect 0 is 1, but there are 3 conjugacy classes of elements g such that the number of solutions $[x, y] = g$ is odd, namley, $g = (1, 2, 3)$, $(1, 2, 3, 4, 5)$, and $(1, 3, 5, 2, 4)$. (B. Sambale, *Letter of 5 May 2021*).

NEW ***12.94.** Let G be a finitely generated pro-p-group not involving the wreath product $C_p \wr \mathbb{Z}_p$ as a closed section (where C_p is a cyclic group of order p and $\mathbb{Z}_p$ is the group of p-adic integers). Does it follow that G is p-adic analytic?

A. Shalev

*No, it does not (A. Jaikin-Zapirain, B. Klopsch, *J. London Math. Soc. (2)*, **76**, no. 2 (2007), 365–383).

NEW **13.39.** Let A be an associative ring with unity and with torsion-free additive group, and let F^A be the tensor product of a free group F by A (A. G. Myasnikov, V. N. Remeslennikov, *Siberian Math. J.*, **35**, no. 5 (1994), 986–996); then F^A is a *free exponential group over A*; in (A. G. Myasnikov, V. N. Remeslennikov, *Int. J. Algebra Comput.*, **6** (1996), 687–711), it is shown how to construct F^A in terms of free products with amalgamation.

*a) (G. Baumslag). Is F^A residually nilpotent torsion-free?

b) Is F^A a linear group?

*c) (G. Baumslag). Is the Magnus homomorphism of $F^{\mathbb{Q}}$ into the group of power series over the rational number field $\mathbb{Q}$ faithful or not?

In the case where $\langle 1 \rangle$ is a pure subgroup of the additive group of A, there are positive answers to questions "a" and "b" (A. M. Gaglione, A. G. Myasnikov, V. N. Remeslennikov, D. Spellman, *Commun. Algebra*, **25** (1997), 631–648). In the same paper it is shown that "a" is equivalent to "c". It is also known that the Magnus homomorphism is one-to-one on any subgroup of $F^{\mathbb{Q}}$ of the type $\langle F, t \mid u = t^n \rangle$ (G. Baumslag, *Commun. Pure Appl. Math.*, **21** (1968), 491–506).

d) Is the universal theory of F^A decidable?

e) (G. Baumslag). Can free A-groups be characterized by a length function?

f) (G. Baumslag). Does a free $\mathbb{Q}$-group admit a free action on some Λ-tree? See definition in (R. Alperin, H. Bass, *in: Combinatorial group theory and topology, Alta, Utah, 1984* (*Ann. Math. Stud.*, **111**), Princeton Univ. Press, 1987, 265–378).

A. G. Myasnikov, V. N. Remeslennikov

*a) Yes, it is (A. Jaikin-Zapirain, *Preprint*, 2020, `http://matematicas.uam.es/~andrei.jaikin/preprints/baumslag.pdf`)

*c) Yes, it is faithful (A. Jaikin-Zapirain, *Preprint*, 2020, `http://matematicas.uam.es/~andrei.jaikin/preprints/baumslag.pdf`).

13.43. (G. R. Robinson). Let G be a finite group and B be a p-block of characters of G. *Conjecture:* If the defect group $D = D(B)$ of the block B is non-abelian, and if $|D : Z(D)| = p^a$, then each character in B has height strictly less than a.

G. R. Robinson's comment of 2020: the conjecture is proved mod CFSG for $p \neq 2$ in (Z. Feng, C. Li, Y. Liu, G. Malle, J. Zhang, *Compos. Math.*, **155**, no. 6 (2019), 1098–1117).

J. Olsson

***14.1.** Suppose that G is a finite group with no non-trivial normal subgroups of odd order, and φ is its 2-automorphism centralizing a Sylow 2-subgroup of G. Is it true that φ^2 is an inner automorphism of G?

R. Zh. Aleev

*Yes, it is true (G. Glauberman, *Math. Z.*, **107** (1968), 1–20).

14.10. *a) (Well-known problem). It is known that any recursively presented group embeds in a finitely presented group (G. Higman, *Proc. Royal Soc. London Ser. A*, **262** (1961), 455–475). Find an explicit and *"natural"* finitely presented group Γ and an embedding of the additive group of the rationals $\mathbb{Q}$ in Γ.

c) Find an explicit and *"natural"* finitely presented group Γ_n and an embedding of $GL_n(\mathbb{Q})$ in Γ_n.

Another phrasing of the same problems is: find a simplicial complex X which covers a finite complex such that the fundamental group of X is $\mathbb{Q}$ or, respectively, $GL_n(\mathbb{Q})$.

P. de la Harpe

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*a) Such an embedding is found in (J. Belk, J. Hyde, F. Matucci, `arXiv:2005.02036`).

NEW **14.53.** *Conjecture:* Let G be a profinite group such that the set of solutions of the equation $x^n = 1$ has positive Haar measure. Then G has an open subgroup H and an element t such that all elements of the coset tH have order dividing n.

This is true in the case $n = 2$. It would be interesting to see whether similar results hold for profinite groups in which the set of solutions of some equation has positive measure.

Comment of 2021: This is also proved for $n = 3$ (A. Abdollahi, M. S. Malekan, `https://arxiv.org/pdf/2012.13886.pdf`).

L. Levai, L. Pyber

***14.60.** Suppose that H is a non-trivial normal subgroup of a finite group G such that the factor-group G/H is isomorphic to one of the simple groups $L_n(q)$, $n \geqslant 3$. Is it true that G has an element whose order is distinct from the order of any element in G/H?

V. D. Mazurov

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*Yes, it is true: for $n \neq 4$ proved in (A. V. Zavarnitsine, *Siberian Math. J.*, **49** (2008), 246–256), and for $n = 4$ in (M. A. Grechkoseeva, S. V. Skresanov, *Siberian Electron. Math. Rep.*, **17** (2020), 585–589). *Editors' comment*: previous claim that it is not true for $n = 4$ was erroneous.

NEW **14.102.** (V. Lin). Let B_n be the braid group on n strings, and let $n > 4$.

a) Does B_n have any non-trivial non-injective endomorphisms?

*b) Is it true that every non-trivial endomorphism of the derived subgroup $[B_n, B_n]$ is an automorphism?

V. Shpilrain

*b) Yes, it is true for $n \geqslant 7$ (K. Kordek, D. Margalit, `https://arxiv.org/abs/1910.06941`).

NEW ***15.15.** Is every maximal subgroup of a finitely generated branch group necessarily of finite index?

L. Bartholdi, R. I. Grigorchuk, Z. Šunić

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*No, not necessarily (I. V. Bondarenko, *Arch. Math. (Basel)*, **95**, no. 4 (2010), 301–308).

NEW ***15.18.** A group is *hereditarily just infinite* if it is residually finite and all of its non-trivial normal subgroups are just infinite.

a) Do there exist finitely generated hereditarily just infinite torsion groups? (It is believed there are none.)

b) Is every finitely generated hereditarily just infinite group necessarily linear?

A positive answer to the question b) would imply a negative answer to a).

L. Bartholdi, R. I. Grigorchuk, Z. Šunić

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*a) Yes, such groups exist (M. Ershov, A. Jaikin-Zapirain, *J. Reine Angew. Math.*, **677** (2013), 71–134).

*b) No, not every (M. Ershov, A. Jaikin-Zapirain, *J. Reine Angew. Math.*, **677** (2013), 71–134).

*15.49. A group G is a *unique product group* if, for any finite nonempty subsets X, Y of G, there is an element of G which can be written in exactly one way in the form xy with $x \in X$ and $y \in Y$. Does there exist a unique product group which is not left-orderable?

P. Linnell

*Yes, it exists (N. Dunfield, *Appendix B* in S. Kionke, J. Raimbault, *Doc. Math.*, **21** (2016), 873–915).

NEW *15.79. Does there exist a Hausdorff group topology on $\mathbb{Z}$ such that the sequence $\{2^n + 3^n\}$ converges to zero?

I. V. Protasov

*Yes, it exists, as follows from Theorem 3 in (I. Z. Ruzsa, *Proc. Conf. Number theory (Budapest, 1987). Vol.* I*: Elementary and analytic*, *Colloq. Math. Soc. János Bolyai*, **51**, North-Holland, Amsterdam, 1990, 473–504). Another solution is in (S. V. Skresanov, *Siberian Math. J.*, **61**, no. 3 (2020), 542–544).

16.15. An element g of a group G is an *Engel element* if for every $h \in G$ there exists k such that $[h, g, \dots, g] = 1$, where g occurs k times; if there is such k independent of h, then g is said to be *boundedly Engel*.

*a) (B. I. Plotkin). Does the set of boundedly Engel elements of a group form a subgroup?

b) The same question for torsion-free groups.

c) The same question for right-ordered groups.

d) The same question for linearly ordered groups.

V. V. Bludov

*No, not always (A. I. Sozutov, *Siberian Math. J.*, **60**, no. 6 (2019) 1099–1100).

*16.42. Is a topological Abelian group (G, τ) compact if every group topology $\tau' \subseteq \tau$ on G is complete? (The answer is yes if every continuous homomorphic image of (G, τ) is complete.)

E. G. Zelenyuk

*Yes, it is (T. Banakh, *Topology Appl.*, **271** (2020), Article ID 106983, 17 p.).

*16.49. Is it true that a free product of groups without generalized torsion is a group without generalized torsion?

V. M. Kopytov, N. Ya. Medvedev

*Yes, it is true; moreover, the generalized torsion in a free product of torsion-free groups is conjugate to a generalized torsion of one of its factor groups (T. Ito, K. Motegi, M. Teragaito, *Proc. Amer. Math. Soc.*, **147**, no. 11 (2019), 4999–5008).

*16.50. Do there exist simple finitely generated right-orderable groups?

There exist finitely generated right-orderable groups coinciding with the derived subgroup (G. Bergman).

V. M. Kopytov, N. Ya. Medvedev

*Yes, such groups do exist (J. Hyde, Y. Lodha, *Invent. Math.*, **218** (2019), 83–112).

NEW **16.53.** Let $d(G)$ denote the smallest cardinality of a generating set of the group G. Suppose that $G = \langle A, B \rangle$, where A and B are two d-generated finite groups of coprime orders. Is it true that $d(G) \leqslant d + 1$?

See Archive 12.71 and (A. Lucchini, *J. Algebra*, **245** (2001), 552–561).

Comment of 2021: the answer is yes if A and B have prime power order (E. Detomi, A. Lucchini, *J. London Math. Soc. (2)*, **87**, no. 3 (2013), 689–706).

A. Lucchini

NEW *16.59. Given a finite group K, does there exist a finite group G such that $K \cong \operatorname{Out} G = \operatorname{Aut} G / \operatorname{Inn} G$? (It is known that an infinite group G exists with this property.)

D. MacHale

*Yes, it does (Y. Cornulier, https://mathoverflow.net/questions/372480/is-every-finite-group-the-outer-automorphism-group-of-a-finite-group/372563).

***16.79.** Is it true that in any finitely generated AT-group over a sequence of cyclic groups of uniformly bounded orders all Sylow subgroups are locally finite? For the definition of an *AT-group* see (A. V. Rozhkov, *Math. Notes*, **40** (1986), 827–836)

A. V. Rozhkov

*No, it is not true (A. V. Rozhkov, in *Group theory and its applications*, Proc. XXII School-Conf. on Group Theory, Kuban' Univ., Krasnodar, 2018, 126–131 (Russian)).

***16.85.** Suppose that groups G, H act faithfully on a regular rooted tree by finite-state automorphisms. Can their free product $G * H$ act faithfully on a regular rooted tree by finite state automorphisms?

V. I. Sushchanskiĭ

*Yes, it can (M. Fedorova, A. Oliynyk, *Algebra Discrete Math.*, **23**, no. 2 (2017), 230–236).

***16.86.** Does the group of all finite-state automorphisms of a regular rooted tree possess an irreducible system of generators?

V. I. Sushchanskiĭ

*Yes, it does (Ya. Lavrenyuk, *Geometriae Dedicata*, **183**, no. 1 (2016), 59–67).

NEW ***16.104.** If G is a finite group, then every element a of the rational group algebra $\mathbb{Q}[G]$ has a unique Jordan decomposition $a = a_s + a_n$, where $a_n \in \mathbb{Q}[G]$ is nilpotent, $a_s \in \mathbb{Q}[G]$ is semisimple over $\mathbb{Q}$, and $a_s a_n = a_n a_s$. The integral group ring $\mathbb{Z}[G]$ is said to have the *additive Jordan decomposition* property (AJD) if a_s, $a_n \in \mathbb{Z}[G]$ for every $a \in \mathbb{Z}[G]$. If $a \in \mathbb{Q}[G]$ is invertible, then a_s is also invertible and so $a = a_s a_u$ with $a_u = 1 + a_s^{-1} a_n$ unipotent and $a_s a_u = a_u a_s$. Such a decomposition is again unique. We say that $\mathbb{Z}[G]$ has *multiplicative Jordan decomposition* property (MJD) if a_s, $a_u \in \mathbb{Z}[G]$ for every invertible $a \in \mathbb{Z}[G]$. See the survey (A. W. Hales, I. B. S. Passi, in: *Algebra, Some Recent Advances*, Birkhäuser, Basel, 1999, 75–87).

Is it true that there are only finitely many isomorphism classes of finite 2-groups G such that $\mathbb{Z}[G]$ has MJD but not AJD?

A. W. Hales, I. B. S. Passi

*Yes, it is true (A. W. Hales, I. B. S. Passi, L. E. Wilson, *J. Algebra*, **316**, no. 1 (2007), 109–132; **371** (2012), 665–666).

***17.3.** Let G be a group in which every 4-element subset contains two elements generating a nilpotent subgroup. Is it true that every 2-generated subgroup of G is nilpotent?

A. Abdollahi

*No, not always (A. I. Sozutov, *Siberian Math. J.*, **60**, no. 6 (2019), 1099–1100).

***17.19.** If F is a free group of finite rank, R a retract of F, and H a subgroup of F of finite rank, must $H \cap R$ be a retract of H?

G. M. Bergman

*No, it must not (I. Snopce, S. Tanushevski, P. Zalesskii, `arXiv:1902.02378`).

***17.20.** If M is a real manifold with nonempty boundary, and G the group of self-homeomorphisms of M which fix the boundary pointwise, is G right-orderable?

G. M. Bergman

*No, not always (J. Hyde, *Ann. of Math. (2)*, **190**, no. 2 (2019), 657–661).

NEW ***17.28.** Is there a soluble right-orderable group with insoluble word problem?

V. V. Bludov, A. M. W. Glass

*Yes, there is (A. Darbinyan, *J. Symbolic Logic*, **85**, no. 4 (2020), 1588–1598).

***17.40.** Let N be a nilpotent subgroup of a finite group G. Do there always exist elements $x, y \in G$ such that $N \cap N^x \cap N^y \leqslant F(G)$? *E. P. Vdovin*

*Yes, such elements always exist, mod CFSG (V. I. Zenkov, *to appear in Siberian Math. J.* (2021)).

17.73. Let G be a finite simple group of Lie type defined over a field of characteristic p, and V an absolutely irreducible G-module over a field of the same characteristic. Is it true that in the cases

*a) $G = U_4(q)$;

. . . .

the split extension of V by G must contain an element whose order is distinct from the order of any element of G? *V. D. Mazurov*

*a) Yes, it is true (M. A. Grechkoseeva, S. V. Skresanov, *Siberian Electron. Math. Rep.*, **17** (2020), 585–589).

***17.108.** Is the group $\langle a, b, t \mid a^t = ab,\ \ b^t = ba\rangle$ linear?

If not, this would be an easy example of a non-linear hyperbolic group. *M. Sapir*

*Yes, this group is linear. Indeed, as noticed by M. Sapir, this group is the mapping torus of an irreducible atoroidal self-monomorphism of a free group; thus it is virtually special, and hence $\mathbb{Z}$-linear by Theorem B in (M. F. Hagen, D. T. Wise, *Duke Math. J.*, **165**, no. 9 (2016), 1753–1813). (M. F. Hagen, *Letter of 6 August 2018.*)

NEW ***18.23.** The normal covering number of the symmetric group S_n of degree n is the minimum number $\gamma(S_n)$ of proper subgroups $H_1, \ldots, H_{\gamma(S_n)}$ of S_n such that every element of S_n is conjugate to an element of H_i, for some $i = 1, \ldots, \gamma(S_n)$. Write $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ for primes $p_1 < \cdots < p_r$ and positive integers $\alpha_1, \ldots, \alpha_r$.

Conjecture:

$$\gamma(S_n) = \begin{cases} \frac{n}{2}\left(1 - \frac{1}{p_1}\right) & \text{if } r = 1 \text{ and } \alpha_1 = 1 \\ \frac{n}{2}\left(1 - \frac{1}{p_1}\right) + 1 & \text{if } r = 1 \text{ and } \alpha_1 \geqslant 2 \\ \frac{n}{2}\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) + 1 & \text{if } r = 2 \text{ and } \alpha_1 + \alpha_2 = 2 \\ \frac{n}{2}\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) + 2 & \text{if } r \geqslant 2 \text{ and } \alpha_1 + \cdots + \alpha_r \geqslant 3 \end{cases}$$

This is the strongest form of the conjecture. We would be also interested in a proof that this holds for n sufficiently large. The result for $r \leqslant 2$, which includes the first three cases above, is proved for n odd (D. Bubboloni, C. E. Praeger, *J. Combin. Theory (A)*, **118** (2011), 2000–2024). When $r \geqslant 3$ we know that $cn \leqslant \gamma(S_n) \leqslant \frac{2}{3}n$ for some positive constant c (D. Bubboloni, C. E. Praeger, P. Spiga, *J. Algebra*, **390** (2013) 199–215). We showed that the conjectured value for $\gamma(S_n)$ is an upper bound, by constructing a normal covering for S_n with this number of conjugacy classes of maximal subgroups, and gave further evidence for the truth of the conjecture in other cases (D. Bubboloni, C. E. Praeger, P. Spiga, *Int. J. Group Theory*, **3**, no. 2 (2014), 57–75).

D. Bubboloni, C. E. Praeger, P. Spiga

*A. Maróti showed that the conjecture is incorrect for odd n; his counterexample is presented in § 2 of (D. Bubboloni, C. E. Praeger, P. Spiga, *Monatsh. Math.*, **191** (2020), 229–247).

***18.31.** Let π be a set of primes. Is it true that in any D_π-group G (see Archive, 3.62) there are three Hall π-subgroups whose intersection coincides with $O_\pi(G)$?

E. P. Vdovin, D. O. Revin

*No, it is not true, for example, for G being an extension of $L_2(27)$ by a field automorphism of order 3, in which H is an extension of a Borel subgroup of $L_2(27)$ by a field automorphism of order 3 (V. I. Zenkov, *Abstracts of Int. Conf. "Mal'tsev Meeting 2020"*, Novosibirsk, 2020, p. 149).

NEW ***18.49.** Let $n \in \mathbb{N}$. Is it true that for any $a, b, c \in \mathbb{N}$ satisfying $1 < a, b, c \leqslant n-2$ the symmetric group S_n has elements of order a and b whose product has order c?

S. Kohl

*Yes, it is (G. A. Miller, *Amer. J. Math.*, **22**, no. 2 (1900), 185–190). Another solution is in (J. König, *Eur. J. Comb.*, **57** (2016), 50–56).

NEW **18.58.** Let G be a group generated by finite number n of involutions in which $(uv)^4 = 1$ for all involutions $u, v \in G$. Is it true that G is finite? is a 2-group? This is true for $n \leqslant 3$.

Editors' comment of 2021: the previously published claim of an affirmative solution of this problem proved to be erroneous.

D. V. Lytkina

NEW **18.73.** a) Does every finitely generated solvable group of derived length $l \geq 2$ embed into a 2-generated solvable group of length $l+1$?

Comment of 2021: It is proved that any countable solvable group of derived length l with torsion-free abelianization embeds in a 2-generated solvable group of derived length $l+1$ (V. A. Roman'kov, *Proc. Amer. Math. Soc.*, **149** (2021), 4133–4143).

*b) Or at least, into some k-generated $(l+1)$-solvable group, where $k = k(l)$?

A. Yu. Ol'shanskiĭ

*b) It is proved that any finitely generated solvable group of derived length l can be embedded in a 4-generated solvable group of derived length $l+1$ (V. A. Roman'kov, *Proc. Amer. Math. Soc.*, **149** (2021), 4133–4143).

NEW **18.98.** The work of many authors shows that most finite simple groups are generated by two elements of orders 2 and 3; for example, see the survey `http://math.nsc.ru/conference/groups2013/slides/MaximVsemirnov_slides.pdf`. Which finite simple groups cannot be generated by two elements of orders 2 and 3?

In particular, is it true that, among classical simple groups of Lie type, such exceptions, apart from $PSU(3, 5^2)$, arise only when the characteristic is 2 or 3?

The question of which finite simple groups are (2,3)-generated remains open only for the orthogonal groups (M. A. Pellegrini, M. C. Tamburini Bellani, `https://arxiv.org/abs/2005.10347`).

M. C. Tamburini

NEW **18.110.** The *non-p-soluble length* of a finite group G is the number of non-p-soluble factors in a shortest normal series each of whose factors either is p-soluble or is a direct product of non-abelian simple groups of order divisible by p. For a given prime p and a given proper group variety $\mathfrak{V}$, is there a bound for the non-p-soluble length of finite groups whose Sylow p-subgroups belong to $\mathfrak{V}$?

Comment of 2021: the existence of such a bound is proved for $p = 2$ (F. Fumagalli, F. Leinen, O. Puglisi, `https://arxiv.org/pdf/2101.09119.pdf`).

E. I. Khukhro, P. Shumyatsky

NEW *__18.115.__ Let G be a finite simple group, and let X, Y be isomorphic simple maximal subgroups of G. Are X and Y conjugate in $\operatorname{Aut} G$?

P. Schmid

*No, not always. For example, there are two $\operatorname{Aut} G$-classes of M_{12} in $E_6(5)$ (P. B. Kleidman, R. A. Wilson, *J. London Math. Soc.*, **42** (1990), 555–561); other counterexamples in $E_6(q)$ for certain q include two classes of $PSL_2(11)$, and two of $PSL_2(19)$ (D. A. Craven, `https://arXiv.org/abs/2103.04869`). (D. A. Craven *Letter of 8 September 2021*).

NEW **19.11.** Does there exist a constant c such that the number of conjugacy classes in a finite group G is always at least $c \log_2 |G|$?

Editors' comment of 2021: It is proved that every group G contains at least $\varepsilon \log |G|/(\log \log |G|)^8$ conjugacy classes for some fixed $\varepsilon > 0$ (L. Pyber, *J. London Math. Soc. (2)*, **46**, no. 2 (1992), 239–249). It is also proved that for every $\varepsilon > 0$ there exists $\delta > 0$ such that every finite group G of order at least 3 has at least $\delta \log_2 |G|/(\log_2 \log_2 |G|)^{3+\varepsilon}$ conjugacy classes (B. Baumeister, A. Maróti, H. P. Tong-Viet, *Forum Math.*, **29**, no. 2 (2017), 259–275).

E. Bertram

*__19.24.__ For a group G, let $\operatorname{Tor}(G)$ be the normal closure of all torsion elements of G. Does there exist a finitely presented group G such that $G/\operatorname{Tor}(G)$ is not finitely presented? Such a group must necessarily be non-hyperbolic.

M. Chiodo, R. Vyas

*Yes, such groups exist: one soluble example and another virtually torsion-free are constructed in (I. J. Leary, A. Minasyan, *Preprint*, 2020).

19.35. Let G be a finite group of order n.

*a) Is it true that for every factorization $n = a_1 \cdots a_k$ there exist subsets $A_1, \ldots, A_k$ such that $|A_1| = a_1, \ldots, |A_k| = a_k$ and $G = A_1 \cdots A_k$?

b) The same question for the case $k = 2$.

M. H. Hooshmand

*a) No, it is not true. A counterexample with $k = 3$ is given by the alternating group on 4 letters $G = A_4$ and $(a_1, a_2, a_3) = (2, 3, 2)$ (G. M. Bergman, *Letter of 19 December 2019*, `https://arxiv.org/pdf/2003.12866.pdf`.)

NEW *__19.36.__ Let G be a periodic group and let $\mathscr{I} = \{x \in G \mid x^2 = 1 \neq x\}$ be the set of its involutions. Let D be a non-empty set of odd integers greater than 1; then G is called a group with D-involutions if $G = \langle \mathscr{I} \rangle$ and for $x, y \in \mathscr{I}$ the order of xy is in the set $\{1, 2\} \cup D$ and all these values actually occur. It is clear that if G is a group with D-involutions, then $\mathscr{I}$ is a single conjugacy class.

Conjecture: If G is a group with $\{3, 5\}$-involutions, then $G \simeq A_5$ or $G \simeq PSU(3, 4)$.

E. Jabara

*The conjecture is proved, even without using the hypothesis that the group G is periodic (E. Bettio, *J. Group Theory*, **24** (2021), 1055–1067).

*__19.37.__ a) Does there exist an absolute constant k such that for any nilpotent injector H of an arbitrary finite group G there are k conjugates of H the intersection of which is equal to the Fitting subgroup $F(G)$ of G?

b) Can one choose $k = 3$ as such a constant? This is true for finite soluble groups (D. S. Passman, *Trans. Amer. Math. Soc.*, **123**, no. 1 (1966), 99–111; A. Mann, *Proc. Amer. Math. Soc.*, **53**, no. 1 (1975), 262–264).

S. F. Kamornikov

*Yes, one can choose $k = 3$ as such a constant, mod CFSG (V. I. Zenkov, *to appear in Siberian Math. J.* (2021)).

***19.40.** Does Thompson's group F (see 12.20) have the Howson property, that is, is the intersection of any two finitely generated subgroups of F finitely generated?

I. Kapovich

*No, it does not. Otherwise any subgroup of F would have this property. But the wreath product $\mathbb{Z}\wr\mathbb{Z}$, which does not have the Howson property (A. S. Kirkinskiĭ, *Algebra Logic*, **20**, no. 1 (1981), 24–36), can be embedded in F (S. Cleary, *Pacific J. Math.*, **228**, no. 1 (2006), 53–61). (D. Robertson, *Letter of 24 April 2018.*)

***19.49.** A *skew brace* is a set B equipped with two operations $+$ and $\cdot$ such that $(B,+)$ is an additively written (but not necessarily abelian) group, $(B,\cdot)$ is a multiplicatively written group, and $a\cdot(b+c)=ab-a+ac$ for any $a,b,c\in B$.

Let A be a skew brace with left-orderable multiplicative group. Is the additive group of A left-orderable?

V. Lebed, L. Vendramin

*No, not always (T. Nasybullov, *J. Algebra*, **540** (2019), 156–167).

***19.50.** A finite graph is said to be integral if all eigenvalues of its adjacency matrix are integers.

a) Let G be a finite group generated by a normal subset R consisting of involutions. Is it true that the Cayley graph $Cay(G,R)$ is integral?

b) Let A_n be the alternating group of degree n, let $S=\{(123),(124),\dots,(12n)\}$ and $R=S\cup S^{-1}$. Is it true that the Cayley graph $Cay(A_n,R)$ is integral?

D. V. Lytkina

*a) Yes, it is true (D. O. Revin, *Letter of 21 April 2018*; see also the reference for part (b) below; A. Abdollahi, *Letter of 3 May 2018*). Both proofs suggested are based on character theory; here is the second one. It suffices to show that the eigenvalues of $Cay(G,R)$ are rational, since the eigenvalues of a simple graph are algebraic integers. It is known that every eigenvalue of $Cay(G,R)$ has the form $\theta_\chi=\frac{1}{\chi(1)}\sum_{r\in R}\chi(r)$ for some complex irreducible character χ of G (implicit on pages 175–177 in P. Diaconis, M. Shahshahani, *Z. Wahrscheinlichkeitstheorie Verw. Gebiete*, **57** (1981), 159–179, see also Theorem 9 in M. R. Murty, *J. Ramanujan Math. Soc.*, **18**, no. 1 (2003) 1–20). Since the value of any complex character on an involution is an integer, it follows that θ_χ is rational.

*b) Yes, it is true (W. Guo, D. V. Lytkina, V. D. Mazurov, D. O. Revin, *Algebra Logic*, **58**, no. 4 (2019), 297–305).

***19.55.** Suppose that in a finite group G every maximal subgroup M is supersoluble whenever $\pi(M)=\pi(G)$, where $\pi(G)$ is the set of all prime divisors of the order of G.

a) What are the non-abelian composition factors of G?

b) Determine the exact upper bounds for the nilpotency length, the rank, and the p-length of G if G is soluble.

V. S. Monakhov

*a) Every nonabelian finite simple group can occur as a composition factors of G (A. Moretó, *Monatsh. Math.*, **195**, no. 3 (2021), 497–500).

*b) There is not any bound for the nilpotency length or the rank, but the p-length is at most 1 for every prime p (A. Moretó, *Monatsh. Math.*, **195**, no. 3 (2021), 497–500).

***19.67.** Let $G\le \mathrm{Sym}(\Omega)$, where Ω is finite. The 2-closure $G^{(2)}$ of the group G is defined to be the largest subgroup of $\mathrm{Sym}(\Omega)$ containing G which has the same orbits as G in the induced action on $\Omega\times\Omega$. Is it true that if G is solvable, then every composition factor of $G^{(2)}$ is either a cyclic or an alternating group?

I. Ponomarenko

*No, it is not true (S. V. Skresanov, *Algebra Logic*, **58**, no. 3 (2019), 249–253).

***19.75.** Let P be a finite 2-group of exponent 2^e such that the rank of every abelian subgroup is at most r. Is it true that $|P| \leq 2^{r(e+1)}$? This bound would be sharp (for a direct product of quaternion groups).

B. Sambale

*No, it is not true, as follows from the examples constructed in (A. Yu. Ol'shanskiĭ, *Math. Notes*, **23** (1978), 183–185). (A. Mann, *Letter of 23 April 2018.*)

NEW ***19.80.** For a periodic group G, let $\pi_e(G)$ denote the set of orders of elements of G. A periodic group G is said to be an *OC_n-group* if $\pi_e(G) = \{1, 2, \ldots, n\}$. Is it true that every OC_7-group is isomorphic to the alternating group A_7?

For finite groups, the answer is affirmative. *W. J. Shi*

*Yes, it is true (E. Jabara, A. S. Mamontov, *Siberian Math. J.*, **62**, no. 1 (2021), 94–104).

NEW ***19.81.** (Well-known problem). Is the conjugacy problem in the braid group B_n in the class NP (that is, decidable in nondeterministic polynomial time with respect to the maximum of the lengths $|u|$, $|v|$, where u, v are the input braid words)? A stronger question: given two conjugate elements of B_n represented by braid words of lengths $\leqslant m$, is there a conjugator whose length is bounded by a polynomial function of m?

V. Shpilrain

*Yes, there is such a conjugator, even for any mapping class groups (J. Tao, *Geom. Funct. Anal.*, **23**, no. 1 (2013), 415–466).

***19.84.** Let $\mathbb{P}$ be the set of all primes, and let $\sigma = \{\sigma_i \mid i \in I\}$ be some partition of $\mathbb{P}$ into disjoint subsets. A finite group G is said to be *σ-primary* if G is a σ_i-group for some i; *σ-nilpotent* if G is a direct product of σ-primary groups; *σ-soluble* if every chief factor of G is σ-primary. A subgroup A of a finite group G is said to be *σ-subnormal in G* if there is a chain $A = A_0 \leqslant A_1 \leqslant \cdots \leqslant A_n = G$ such that for every i either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_{i-1})_{A_i}$ is σ-primary, where $(A_{i-1})_{A_i}$ is the largest normal subgroup of A_i contained in A_{i-1}.

Suppose that a subgroup A of a finite group G is σ-subnormal in $\langle A, A^x\rangle$ for all $x \in G$. Is it true that then A is σ-subnormal in G?

An affirmative answer is known if $\sigma = \{\{2\}, \{3\}, \ldots\}$ (Wielandt). *A. N. Skiba*

*No, not always. For example, in $G = S_5$ with partition $\sigma = \{2, 3\} \cup \{5\}$ the subgroup $A = \langle(12)\rangle$ is σ-subnormal in $\langle A, A^x\rangle$ for every $x \in G$, but it is not σ-subnormal in G. (V. N. Tyutyanov, *Letter of 28 August 2019.*)

NEW ***19.85.** Suppose that every Schmidt subgroup of a finite group G is σ-subnormal in G (see 19.84). Is it true that then there is a normal σ-nilpotent subgroup $N \leqslant G$ such that G/N is cyclic?

An affirmative answer is known if $\sigma = \{\{2\}, \{3\}, \ldots\}$. *A. N. Skiba*

*Yes, it is true (X. Yi, S. F. Kamornikov, *J. Algebra*, **560** (2020), 181–191).

***19.87.** Suppose that for every Sylow subgroup P of a finite group G and every maximal subgroup V of P there is a σ-soluble subgroup T such that $VT = G$. Is it true that then G is σ-soluble?

A. N. Skiba

*Yes, it is true (A.-M. Liu, W. Guo, I. N. Safonova, A. N. Skiba, *to appear in J. Algebra*, 2021); another solution using CFSG is in (S. F. Kamornikov, V. N. Tyutyanov, *Trudy Inst. Mat. Mekh. UrO RAN*, **27**, no. 1 (2021), 98–102 (Russian); X. Yi, S. F. Kamornikov, V. N. Tyutyanov, *Probl. Physics, Math. Techn.*, **46**, no. 1 (2021), 50–53).

*****19.88.** Suppose that for every Sylow subgroup P of a finite group G and every maximal subgroup V of P there is a σ-nilpotent subgroup T such that $VT = G$. Is it true that then G is σ-nilpotent?

A. N. Skiba

*Yes, it is true (A.-M. Liu, W. Guo, I. N. Safonova, A. N. Skiba, *to appear in J. Algebra*, 2021); another solution using CFSG is in (S. F. Kamornikov, V. N. Tyutyanov, *Trudy Inst. Mat. Mekh. UrO RAN*, **27**, no. 1 (2021), 98–102 (Russian); X. Yi, S. F. Kamornikov, V. N. Tyutyanov, *Probl. Physics, Math. Techn.*, **46**, no. 1 (2021), 50–53).

19.90. A *skew brace* is a set B equipped with two operations $+$ and $\cdot$ such that $(B, +)$ is an additively written (but not necessarily abelian) group, $(B, \cdot)$ is a multiplicatively written group, and $a \cdot (b + c) = ab - a + ac$ for any $a, b, c \in B$.

*a) Is there a skew brace with soluble additive group but non-soluble multiplicative group?

b) Is there a skew brace with non-soluble additive group but nilpotent multiplicative group?

c) Is there a finite skew brace with soluble additive group but non-soluble multiplicative group?

*d) Is there a finite skew brace with non-soluble additive group but nilpotent multiplicative group?

A. Smoktunowicz, L. Vendramin

*a) Yes, there is (T. Nasybullov, *J. Algebra*, **540** (2019), 156–167).

*d) No, there is not (C. Tsang, Q. Chao, *Int. J. Algebra Comput.*, **30**, no. 2 (2020), 253–265).

NEW ***19.98.** A connected graph Σ is a *symmetrical extension* of a graph Γ by a graph Δ if there exist a vertex-transitive group G of automorphisms of Σ and an imprimitivity system σ of G on the set of vertices of Σ such that the quotient graph Σ/σ is isomorphic to Γ and blocks of σ generate in Σ subgraphs isomorphic to Δ.

(a) Let Γ be a locally finite Cayley graph of a finitely presented group, and Δ a finite graph. Are there only finitely many symmetrical extensions of Γ by Δ?

(b) Let Γ be a locally finite graph which has the property of k-contractibility for some positive integer k (see the definition in (V. I. Trofimov, *Proc. Steklov Inst. Math.*, **279**, suppl. 1 (2012), 107–112); note that any Γ from (a) is such a graph) and let Δ be a finite graph. Are there only finitely many symmetrical extensions of Γ by Δ?

V. I. Trofimov

*(a) No, not always, as follows from the construction in the proof of Theorem H in (M. de la Salle, R. Tessera, *J. Topology*, **12** (2019), 705–743).

*(b) No, not always as follows from the construction in the proof of Theorem H in (M. de la Salle, R. Tessera, *J. Topology*, **12** (2019), 705–743).

***19.101.** The maximum length of a chain of nested centralizers of a group is called its c-dimension. Let G be a locally finite group of finite c-dimension k, and let S be the preimage in G of the socle of G/R, where R is the locally solvable radical of G. Is it true that the factor group G/S contains an abelian subgroup of index bounded by a function of k?

A. V. Vasil'ev

*Yes, it is true (A. A. Buturlakin, *J. Algebra Appl.*, **18**, no. 12 (2019), 1950223, 12 pp).

*__19.109.__ A subgroup H of a finite group G is called pronormal if for any $g \in G$ the subgroups H and H^g are conjugate by an element of $\langle H, H^g \rangle$. A maximal subgroup of a maximal subgroup is called second maximal. Is it true that in a non-abelian finite simple group G all maximal subgroups are Hall subgroups if and only if every second maximal subgroup of G is pronormal in G?

V. I. Zenkov

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*No, not always, for example, in $SL_2(2^{11})$ every second maximal subgroup is pronormal (V. N. Tyutyanov, *Letter of 23 November 2018*).