December 2020 update

See below the latest updates for the current 19th issue, which are all incorporated in the PDF file of the Kourovka Notebook available at `https://kourovka-notebook.org` (see also `http://math.nsc.ru/~alglog/19tkt.pdf` and `https://arxiv.org/pdf/1401.0300.pdf`).

For convenience of the readers all the updates since the first appearance of the 19th edition are also listed below; the newest ones added in December 2020 are marked **NEW**.

***3.51.** Is it true that every finite group with a group of automorphisms Φ which acts regularly on the set of conjugacy classes of G (that is, leaves only the identity class fixed) is soluble? The answer is known to be affirmative in the case where Φ is a cyclic group generated by a regular automorphism.

A. I. Saksonov

*No, not always (Y. Fine, *J. Group Theory*, **22**, no. 6, 1077–1087).

***8.31.** Describe the finite groups in which every proper subgroup has a complement in some larger subgroup. Among these groups are, for example, $PSL_2(7)$ and all Sylow subgroups of symmetric groups.

V. M. Levchuk

*These groups are described independently in (V. M. Levchuk, A. G. Likharev, *Siberian Math. J.*, **47**, no. 4 (2006), 659–668) and in (V. N. Tyutyanov, *Proc. Gomel' State Univ.*, **3** (2006), 178–183 (in Russian)).

***12.32.** Prove an analogue of Higman's theorem for the Burnside variety $\mathfrak{B}_n$ of groups of odd exponent $n \gg 1$, that is, prove that every recursively presented group of exponent n can be embedded in a finitely presented (in $\mathfrak{B}_n$) group of exponent n.

S. V. Ivanov

*It is proved (A. Olshanskii, *to appear in J. Algebra*, `arXiv:1909.10113`).

***12.38.** (J. G. Thompson). For a finite group G, we denote by $N(G)$ the set of all orders of the conjugacy classes of G. Is it true that if G is a finite non-abelian simple group, H a finite group with trivial centre and $N(G) = N(H)$, then G and H are isomorphic?

A. S. Kondratiev, W. J. Shi

*Yes, it is true. The final step of the proof is in the paper (I. B. Gorshkov, *Commun. Algebra*, **47**, no. 12 (2019), 5192–5206), which contains references to the previous steps by M. Ahanjideh, N. Ahanjideh, S. H. Alavi, G. Y. Chen, A. Daneshkhah, M. R. Darafsheh, I. B. Gorshkov, A. Iranmanesh, I. Kaygorodov, Behn. Khosravi, Behr. Khosravi, A. Kukharev, W. Shi, A. Shlepkin, A. V. Vasil'ev, L. Wang, M. Xu.

***12.79.** Suppose that a and b are two elements of a finite group G such that the function

$$\varphi(g) = 1^G(g) - 1^G_{\langle a\rangle}(g) - 1^G_{\langle b\rangle}(g) - 1^G_{\langle ab\rangle}(g) + 2$$

is a character of G. Is it true that $G = \langle a, b\rangle$? The converse statement is true.

S. P. Strunkov

*No, not always: a counterexample is given by $G = A_4$, $a = b = (123)$. (S. V. Skresanov, *Algebra Logic*, **58**, no. 3 (2019), 249–253).

13.43. (G. R. Robinson). Let G be a finite group and B be a p-block of characters of G. *Conjecture:* If the defect group $D = D(B)$ of the block B is non-abelian, and if $|D : Z(D)| = p^a$, then each character in B has height strictly less than a.

NEW *G. R. Robinson's comment of 2020*: the conjecture is proved mod CFSG for $p \neq 2$ in (Z. Feng, C. Li, Y. Liu, G. Malle, J. Zhang, *Compos. Math.*, **155**, no. 6 (2019), 1098–1117).

J. Olsson

***14.1.** Suppose that G is a finite group with no non-trivial normal subgroups of odd order, and φ is its 2-automorphism centralizing a Sylow 2-subgroup of G. Is it true that φ^2 is an inner automorphism of G?

R. Zh. Aleev

*Yes, it is true (G. Glauberman, *Math. Z.*, **107** (1968), 1–20).

***14.60.** Suppose that H is a non-trivial normal subgroup of a finite group G such that the factor-group G/H is isomorphic to one of the simple groups $L_n(q)$, $n \geqslant 3$. Is it true that G has an element whose order is distinct from the order of any element in G/H?

V. D. Mazurov

*Yes, it is true: for $n \neq 4$ proved in (A. V. Zavarnitsine, *Siberian Math. J.*, **49** (2008), 246–256), and for $n = 4$ in (M. A. Grechkoseeva, S. V. Skresanov, *Siberian Electron. Math. Rep.*, **17** (2020), 585–589). *Editors' comment*: previous claim that it is not true for $n = 4$ was erroneous.

***15.49.** A group G is a *unique product group* if, for any finite nonempty subsets X, Y of G, there is an element of G which can be written in exactly one way in the form xy with $x \in X$ and $y \in Y$. Does there exist a unique product group which is not left-orderable?

P. Linnell

*Yes, it exists (N. Dunfield, *Appendix B* in S. Kionke, J. Raimbault, *Doc. Math.*, **21** (2016), 873–915).

***15.79.** Does there exist a Hausdorff group topology on $\mathbb{Z}$ such that the sequence $\{2^n + 3^n\}$ converges to zero?

I. V. Protasov

*Yes, it exists (S. V. Skresanov, *Siberian Math. J.*, **61**, no. 3 (2020), 542–544).

16.15. An element g of a group G is an *Engel element* if for every $h \in G$ there exists k such that $[h, g, \ldots, g] = 1$, where g occurs k times; if there is such k independent of h, then g is said to be *boundedly Engel*.

*a) (B. I. Plotkin). Does the set of boundedly Engel elements of a group form a subgroup?

b) The same question for torsion-free groups.

c) The same question for right-ordered groups.

d) The same question for linearly ordered groups.

V. V. Bludov

*No, not always (A. I. Sozutov, *Siberian Math. J.*, **60**, no. 6 (2019) 1099–1100).

***16.42.** Is a topological Abelian group (G, τ) compact if every group topology $\tau' \subseteq \tau$ on G is complete? (The answer is yes if every continuous homomorphic image of (G, τ) is complete.)

E. G. Zelenyuk

*Yes, it is (T. Banakh, `arXiv.org/abs/1706.05411`).

***16.49.** Is it true that a free product of groups without generalized torsion is a group without generalized torsion?

V. M. Kopytov, N. Ya. Medvedev

*Yes, it is true; moreover, the generalized torsion in a free product of torsion-free groups is conjugate to a generalized torsion of one of its factor groups (T. Ito, K. Motegi, M. Teragaito, *Proc. Amer. Math. Soc.*, **147**, no. 11 (2019), 4999–5008).

***16.50.** Do there exist simple finitely generated right-orderable groups?

There exist finitely generated right-orderable groups coinciding with the derived subgroup (G. Bergman).

V. M. Kopytov, N. Ya. Medvedev

*Yes, such groups do exist (J. Hyde, Y. Lodha, *Invent. Math.*, **218** (2019), 83–112).

***16.79.** Is it true that in any finitely generated AT-group over a sequence of cyclic groups of uniformly bounded orders all Sylow subgroups are locally finite? For the definition of an *AT-group* see (A. V. Rozhkov, *Math. Notes*, **40** (1986), 827–836)

A. V. Rozhkov

*No, it is not true (A. V. Rozhkov, in *Group theory and its applications*, Proc. XXII School-Conf. on Group Theory, Kuban' Univ., Krasnodar, 2018, 126–131 (Russian)).

***16.85.** Suppose that groups G, H act faithfully on a regular rooted tree by finite-state automorphisms. Can their free product $G * H$ act faithfully on a regular rooted tree by finite state automorphisms?

V. I. Sushchanskiĭ

*Yes, it can (M. Fedorova, A. Oliynyk, *Algebra Discrete Math.*, **23**, no. 2 (2017), 230–236).

***16.86.** Does the group of all finite-state automorphisms of a regular rooted tree possess an irreducible system of generators?

V. I. Sushchanskiĭ

*Yes, it does (Ya. Lavrenyuk, *Geometriae Dedicata*, **183**, no. 1 (2016), 59–67).

***17.3.** Let G be a group in which every 4-element subset contains two elements generating a nilpotent subgroup. Is it true that every 2-generated subgroup of G is nilpotent?

A. Abdollahi

*No, not always (A. I. Sozutov, *Siberian Math. J.*, **60**, no. 6 (2019), 1099–1100).

***17.19.** If F is a free group of finite rank, R a retract of F, and H a subgroup of F of finite rank, must $H \cap R$ be a retract of H?

G. M. Bergman

*No, it must not (I. Snopce, S. Tanushevski, P. Zalesskii, `arXiv:1902.02378`).

***17.20.** If M is a real manifold with nonempty boundary, and G the group of self-homeomorphisms of M which fix the boundary pointwise, is G right-orderable?

G. M. Bergman

*No, not always (J. Hyde, *Ann. of Math. (2)*, **190**, no. 2 (2019), 657–661).

17.40. Let N be a nilpotent subgroup of a finite group G. Do there always exist elements $x, y \in G$ such that $N \cap N^x \cap N^y \leqslant F(G)$?

Editors' comment (2018): An affirmative solution is announced in (V. I. Zenkov, *Abstracts of Int. Conf. "Mal'tsev Meeting 2018"*, Novosibirsk, 2018, p. 94).

E. P. Vdovin

17.73. Let G be a finite simple group of Lie type defined over a field of characteristic p, and V an absolutely irreducible G-module over a field of the same characteristic. Is it true that in the cases

*a) $G = U_4(q)$;

. . . .

the split extension of V by G must contain an element whose order is distinct from the order of any element of G?

V. D. Mazurov

*a) Yes, it is true (M. A. Grechkoseeva, S. V. Skresanov, *Siberian Electron. Math. Rep.*, **17** (2020), 585–589).

***17.108.** Is the group $\langle a, b, t \mid a^t = ab,\ b^t = ba \rangle$ linear?

If not, this would be an easy example of a non-linear hyperbolic group. *M. Sapir*

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*Yes, this group is linear. Indeed, as noticed by M. Sapir, this group is the mapping torus of an irreducible atoroidal self-monomorphism of a free group; thus it is virtually special, and hence $\mathbb{Z}$-linear by Theorem B in (M. F. Hagen, D. T. Wise, *Duke Math. J.*, **165**, no. 9 (2016), 1753–1813). (M. F. Hagen, *Letter of 6 August 2018.*)

***18.31.** Let π be a set of primes. Is it true that in any D_π-group G (see Archive, 3.62) there are three Hall π-subgroups whose intersection coincides with $O_\pi(G)$?

E. P. Vdovin, D. O. Revin

NEW *No, it is not true, for example, for G being an extension of $L_2(27)$ by a field automorphism of order 3, in which H is an extension of a Borel subgroup of $L_2(27)$ by a field automorphism of order 3 (V. I. Zenkov, *Abstracts of Int. Conf. "Mal'tsev Meeting 2020"*, Novosibirsk, 2020, p. 149).

***18.58.** Let G be a group generated by finite number n of involutions in which $(uv)^4 = 1$ for all involutions $u, v \in G$. Is it true that G is finite? is a 2-group? This is true for $n \le 3$. *D. V. Lytkina*

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*Yes, it is a finite 2-group (E. Jabara, *J. Group Theory*, **19** (2016), 705–711).

18.73. Does every finitely generated solvable group of derived length $l \ge 2$ embed into a 2-generated solvable group of length $l+1$? Or at least, into some k-generated $(l+1)$-solvable group, where $k = k(l)$?

NEW *Editors' comment (2020)*: It is announced in (V. A. Roman'kov, *Abstracts of Int. Conf. "Mal'tsev Meeting 2020"*, Novosibirsk, 2020, p. 30) that any finitely generated solvable group G of derived length l can be embedded in a 4-generated solvable group H of derived length $l+1$. *A. Yu. Ol'shanskiĭ*

***19.24.** For a group G, let $\mathrm{Tor}(G)$ be the normal closure of all torsion elements of G. Does there exist a finitely presented group G such that $G/\mathrm{Tor}(G)$ is not finitely presented? Such a group must necessarily be non-hyperbolic.

M. Chiodo, R. Vyas

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*Yes, such groups exist: one soluble example and another virtually torsion-free are constructed in (I. J. Leary, A. Minasyan, *Preprint*, 2020).

19.35. Let G be a finite group of order n.

*a) Is it true that for every factorization $n = a_1 \cdots a_k$ there exist subsets $A_1, \ldots, A_k$ such that $|A_1| = a_1, \ldots, |A_k| = a_k$ and $G = A_1 \cdots A_k$?

b) The same question for the case $k = 2$. *M. H. Hooshmand*

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*a) No, it is not true. A counterexample with $k = 3$ is given by the alternating group on 4 letters $G = A_4$ and $(a_1, a_2, a_3) = (2, 3, 2)$ (G. M. Bergman, *Letter of 19 December 2019*, `https://arxiv.org/pdf/2003.12866.pdf`.)

***19.40.** Does Thompson's group F (see 12.20) have the Howson property, that is, is the intersection of any two finitely generated subgroups of F finitely generated?

I. Kapovich

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*No, it does not. Otherwise any subgroup of F would have this property. But the wreath product $\mathbb{Z} \wr \mathbb{Z}$, which does not have the Howson property (A. S. Kirkinskiĭ, *Algebra Logic*, **20**, no. 1 (1981), 24–36), can be embedded in F (S. Cleary, *Pacific J. Math.*, **228**, no. 1 (2006), 53–61). (D. Robertson, *Letter of 24 April 2018.*)

***19.49.** A *skew brace* is a set B equipped with two operations $+$ and $\cdot$ such that $(B,+)$ is an additively written (but not necessarily abelian) group, $(B,\cdot)$ is a multiplicatively written group, and $a\cdot(b+c)=ab-a+ac$ for any $a,b,c\in B$.

Let A be a skew brace with left-orderable multiplicative group. Is the additive group of A left-orderable?

V. Lebed, L. Vendramin

*No, not always (T. Nasybullov, *J. Algebra*, **540** (2019), 156–167).

***19.50.** A finite graph is said to be integral if all eigenvalues of its adjacency matrix are integers.

a) Let G be a finite group generated by a normal subset R consisting of involutions. Is it true that the Cayley graph $Cay(G,R)$ is integral?

b) Let A_n be the alternating group of degree n, let $S=\{(123),(124),\dots,(12n)\}$ and $R=S\cup S^{-1}$. Is it true that the Cayley graph $Cay(A_n,R)$ is integral?

D. V. Lytkina

*a) Yes, it is true (D. O. Revin, *Letter of 21 April 2018*; see also the reference for part (b) below; A. Abdollahi, *Letter of 3 May 2018*). Both proofs suggested are based on character theory; here is the second one. It suffices to show that the eigenvalues of $Cay(G,R)$ are rational, since the eigenvalues of a simple graph are algebraic integers. It is known that every eigenvalue of $Cay(G,R)$ has the form $\theta_\chi=\frac{1}{\chi(1)}\sum_{r\in R}\chi(r)$ for some complex irreducible character χ of G (implicit on pages 175–177 in P. Diaconis, M. Shahshahani, *Z. Wahrscheinlichkeitstheorie Verw. Gebiete*, **57** (1981), 159–179, see also Theorem 9 in M. R. Murty, *J. Ramanujan Math. Soc.*, **18**, no. 1 (2003) 1–20). Since the value of any complex character on an involution is an integer, it follows that θ_χ is rational.

*b) Yes, it is true (W. Guo, D. V. Lytkina, V. D. Mazurov, D. O. Revin, *Algebra Logic*, **58**, no. 4 (2019), 297–305).

NEW ***19.55.** Suppose that in a finite group G every maximal subgroup M is supersoluble whenever $\pi(M)=\pi(G)$, where $\pi(G)$ is the set of all prime divisors of the order of G.

a) What are the non-abelian composition factors of G?

b) Determine the exact upper bounds for the nilpotency length, the rank, and the p-length of G if G is soluble.

V. S. Monakhov

*a) Every nonabelian finite simple group can occur as a composition factors of G (A. Moretó, *to appear in Monatsh. Math.*, `http://arxiv.org/abs/2010.04808`).

*b) There is not any bound for the nilpotency length or the rank, but the p-length is at most 1 for every prime p (A. Moretó, *to appear in Monatsh. Math.*, `http://arxiv.org/abs/2010.04808`).

***19.67.** Let $G\le \mathrm{Sym}(\Omega)$, where Ω is finite. The 2-closure $G^{(2)}$ of the group G is defined to be the largest subgroup of $\mathrm{Sym}(\Omega)$ containing G which has the same orbits as G in the induced action on $\Omega\times\Omega$. Is it true that if G is solvable, then every composition factor of $G^{(2)}$ is either a cyclic or an alternating group?

I. Ponomarenko

*No, it is not true (S. V. Skresanov, *Algebra Logic*, **58**, no. 3 (2019), 249–253).

***19.75.** Let P be a finite 2-group of exponent 2^e such that the rank of every abelian subgroup is at most r. Is it true that $|P|\le 2^{r(e+1)}$? This bound would be sharp (for a direct product of quaternion groups).

B. Sambale

*No, it is not true, as follows from the examples constructed in (A. Yu. Ol'shanskiĭ, *Math. Notes*, **23** (1978), 183–185). (A. Mann, *Letter of 23 April 2018*.)

***19.84.** Let $\mathbb{P}$ be the set of all primes, and let $\sigma = \{\sigma_i \mid i \in I\}$ be some partition of $\mathbb{P}$ into disjoint subsets. A finite group G is said to be *σ-primary* if G is a σ_i-group for some i; *σ-nilpotent* if G is a direct product of σ-primary groups; *σ-soluble* if every chief factor of G is σ-primary. A subgroup A of a finite group G is said to be *σ-subnormal in G* if there is a chain $A = A_0 \leqslant A_1 \leqslant \cdots \leqslant A_n = G$ such that for every i either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_{i-1})_{A_i}$ is σ-primary, where $(A_{i-1})_{A_i}$ is the largest normal subgroup of A_i contained in A_{i-1}.

Suppose that a subgroup A of a finite group G is σ-subnormal in $\langle A, A^x\rangle$ for all $x \in G$. Is it true that then A is σ-subnormal in G?

An affirmative answer is known if $\sigma = \{\{2\}, \{3\}, \ldots\}$ (Wielandt). *A. N. Skiba*

*No, not always. For example, in $G = S_5$ with partition $\sigma = \{2,3\} \cup \{5\}$ the subgroup $A = \langle (12)\rangle$ is σ-subnormal in $\langle A, A^x\rangle$ for every $x \in G$, but it is not σ-subnormal in G. (V. N. Tyutyanov, *Letter of 28 August 2019.*)

19.90. A *skew brace* is a set B equipped with two operations $+$ and $\cdot$ such that $(B, +)$ is an additively written (but not necessarily abelian) group, $(B, \cdot)$ is a multiplicatively written group, and $a \cdot (b + c) = ab - a + ac$ for any $a, b, c \in B$.

*a) Is there a skew brace with soluble additive group but non-soluble multiplicative group?

b) Is there a skew brace with non-soluble additive group but nilpotent multiplicative group?

c) Is there a finite skew brace with soluble additive group but non-soluble multiplicative group?

*d) Is there a finite skew brace with non-soluble additive group but nilpotent multiplicative group? *A. Smoktunowicz, L. Vendramin*

*a) Yes, there is (T. Nasybullov, *J. Algebra*, **540** (2019), 156–167).

*d) No, there is not (C. Tsang, Q. Chao, `arXiv:1901.10636`).

***19.101.** The maximum length of a chain of nested centralizers of a group is called its c-dimension. Let G be a locally finite group of finite c-dimension k, and let S be the preimage in G of the socle of G/R, where R is the locally solvable radical of G. Is it true that the factor group G/S contains an abelian subgroup of index bounded by a function of k? *A. V. Vasil'ev*

*Yes, it is true (A. A. Buturlakin, *J. Algebra Appl.*, **18**, no. 12 (2019), 1950223, 12 pp).

***19.109.** A subgroup H of a finite group G is called pronormal if for any $g \in G$ the subgroups H and H^g are conjugate by an element of $\langle H, H^g\rangle$. A maximal subgroup of a maximal subgroup is called second maximal. Is it true that in a non-abelian finite simple group G all maximal subgroups are Hall subgroups if and only if every second maximal subgroup of G is pronormal in G? *V. I. Zenkov*

*No, not always, for example, in $SL_2(2^{11})$ every second maximal subgroup is pronormal (V. N. Tyutyanov, *Letter of 23 November 2018*).