December 2018 update

See below the latest updates for the current 19th issue, which are all incorporated in the PDF file of the Kourovka Notebook available at `https://kourovka-notebook.org` (see also `http://math.nsc.ru/~alglog/19tkt.pdf` and `https://arxiv.org/pdf/1401.0300.pdf`).

For convenience of the readers all the updates since the first appearance of the 19th edition are also listed below; the newest ones added in December 2018 are marked **NEW**.

***12.79.** Suppose that a and b are two elements of a finite group G such that the function

$$\varphi(g) = 1^G(g) - 1^G_{\langle a\rangle}(g) - 1^G_{\langle b\rangle}(g) - 1^G_{\langle ab\rangle}(g) + 2$$

is a character of G. Is it true that $G = \langle a, b\rangle$? The converse statement is true.

S. P. Strunkov

*No, not always: a counterexample is given by $G = A_4$, $a = b = (123)$. (S. V. Skresanov, *Letter of 20 July 2018*).

16.15. An element g of a group G is an *Engel element* if for every $h \in G$ there exists k such that $[h, g, \dots, g] = 1$, where g occurs k times; if there is such k independent of h, then g is said to be *boundedly Engel.*

NEW *a) (B. I. Plotkin). Does the set of boundedly Engel elements of a group form a subgroup?

b) The same question for torsion-free groups.

c) The same question for right-ordered groups.

d) The same question for linearly ordered groups. *V. V. Bludov*

*No, not always (A. I. Sozutov, *Preprint*, 2018, submitted to *Siberian Math. J.*).

***16.79.** Is it true that in any finitely generated AT-group over a sequence of cyclic groups of uniformly bounded orders all Sylow subgroups are locally finite? For the definition of an *AT-group* see (A. V. Rozhkov, *Math. Notes*, **40** (1986), 827–836)

A. V. Rozhkov

*No, it is not true (A. V. Rozhkov, in *Group theory and its applications*, Proc. XXII School-Conf. on Group Theory, Kuban' Univ., Krasnodar, 2018, 126–131 (Russian)).

***16.85.** Suppose that groups G, H act faithfully on a regular rooted tree by finite-state automorphisms. Can their free product $G * H$ act faithfully on a regular rooted tree by finite state automorphisms? *V. I. Sushchanskiĭ*

*Yes, it can (M. Fedorova, A. Oliynyk, *Algebra Discrete Math.*, **23**, no. 2 (2017), 230–236).

***16.86.** Does the group of all finite-state automorphisms of a regular rooted tree possess an irreducible system of generators? *V. I. Sushchanskiĭ*

*Yes, it does (Ya. Lavrenyuk, *Geometriae Dedicata*, **183**, no. 1 (2016), 59–67).

NEW ***17.3.** Let G be a group in which every 4-element subset contains two elements generating a nilpotent subgroup. Is it true that every 2-generated subgroup of G is nilpotent? *A. Abdollahi*

*No, not always (A. I. Sozutov, *Preprint*, 2018, submitted to *Siberian Math. J.*).

NEW *__17.20.__ If M is a real manifold with nonempty boundary, and G the group of self-homeomorphisms of M which fix the boundary pointwise, is G right-orderable?

G. M. Bergman

*No, not always (J. Hyde, `https://arxiv.org/abs/1810.12851`).

17.40. Let N be a nilpotent subgroup of a finite group G. Do there always exist elements $x, y \in G$ such that $N \cap N^x \cap N^y \leqslant F(G)$?

NEW *Editors' comment (2018)*: An affirmative solution is announced in (V. I. Zenkov, Abstracts of Int. Conf. "Mal'tsev Meeting 2018", Novosibirsk, 2018, p. 94).

E. P. Vdovin

*__17.108.__ Is the group $\langle a, b, t \mid a^t = ab,\ \ b^t = ba \rangle$ linear?

If not, this would be an easy example of a non-linear hyperbolic group. *M. Sapir*

*Yes, this group is linear. Indeed, as noticed by M. Sapir, this group is the mapping torus of an irreducible atoroidal self-monomorphism of a free group; thus it is virtually special, and hence $\mathbb{Z}$-linear by Theorem B in (M. F. Hagen, D. T. Wise, *Duke Math. J.*, **165**, no. 9 (2016), 1753–1813). (M. F. Hagen, *Letter of 6 August 2018.*)

*__19.40.__ Does Thompson's group F (see 12.20) have the Howson property, that is, is the intersection of any two finitely generated subgroups of F finitely generated?

I. Kapovich

*No, it does not. Otherwise any subgroup of F would have this property. But the wreath product $\mathbb{Z} \wr \mathbb{Z}$, which does not have the Howson property (A. S. Kirkinskiĭ, *Algebra Logic*, **20**, no. 1 (1981), 24–36), can be embedded in F (S. Cleary, *Pacific J. Math.*, **228**, no. 1 (2006), 53–61). (D. Robertson, *Letter of 24 April 2018.*)

NEW *__19.49.__ A *skew brace* is a set B equipped with two operations $+$ and $\cdot$ such that $(B, +)$ is an additively written (but not necessarily abelian) group, $(B, \cdot)$ is a multiplicatively written group, and $a \cdot (b + c) = ab - a + ac$ for any $a, b, c \in B$.

Let A be a skew brace with left-orderable multiplicative group. Is the additive group of A left-orderable?

V. Lebed, L. Vendramin

*No, not always (T. Nasybullov, `arXiv:1809.09418`).

*__19.50.__ A finite graph is said to be integral if all eigenvalues of its adjacency matrix are integers.

a) Let G be a finite group generated by a normal subset R consisting of involutions. Is it true that the Cayley graph $Cay(G, R)$ is integral?

b) Let A_n be the alternating group of degree n, let $S = \{(123), (124), \ldots, (12n)\}$ and $R = S \cup S^{-1}$. Is it true that the Cayley graph $Cay(A_n, R)$ is integral?

D. V. Lytkina

*a) Yes, it is true (D. O. Revin, *Letter of 21 April 2018*; see also the reference for part (b) below; A. Abdollahi, *Letter of 3 May 2018*). Both proofs suggested are based on character theory; here is the second one. It suffices to show that the eigenvalues of $Cay(G, R)$ are rational, since the eigenvalues of a simple graph are algebraic integers. It is known that every eigenvalue of $Cay(G, R)$ has the form $\theta_\chi = \frac{1}{\chi(1)} \sum_{r \in R} \chi(r)$ for some complex irreducible character χ of G (implicit on pages 175–177 in P. Diaconis, M. Shahshahani, *Z. Wahrscheinlichkeitstheorie Verw. Gebiete*, **57** (1981), 159–179, see also Theorem 9 in M. R. Murty, *J. Ramanujan Math. Soc.*, **18**, no. 1 (2003) 1–20). Since the value of any complex character on an involution is an integer, it follows that θ_χ is rational.

*b) Yes, it is true (W. Guo, D. V. Lytkina, V. D. Mazurov, D. O. Revin, *Preprint*, 2019, `arXiv:1808.01391`).

***19.75.** Let P be a finite 2-group of exponent 2^e such that the rank of every abelian subgroup is at most r. Is it true that $|P| \leq 2^{r(e+1)}$? This bound would be sharp (for a direct product of quaternion groups).

B. Sambale

*No, it is not true, as follows from the examples constructed in (A. Yu. Ol'shanskiĭ, *Math. Notes*, **23** (1978), 183–185). (A. Mann, *Letter of 23 April 2018.*)

19.90. A *skew brace* is a set B equipped with two operations $+$ and $\cdot$ such that $(B, +)$ is an additively written (but not necessarily abelian) group, $(B, \cdot)$ is a multiplicatively written group, and $a \cdot (b + c) = ab - a + ac$ for any $a, b, c \in B$.

NEW *a) Is there a skew brace with soluble additive group but non-soluble multiplicative group?

b) Is there a skew brace with non-soluble additive group but nilpotent multiplicative group?

NEW c) Is there a finite skew brace with soluble additive group but non-soluble multiplicative group?

NEW d) Is there a finite skew brace with non-soluble additive group but nilpotent multiplicative group?

A. Smoktunowicz, L. Vendramin

*a) Yes, there is (T. Nasybullov, `arXiv:1809.09418`).

***19.101.** The maximum length of a chain of nested centralizers of a group is called its *c*-dimension. Let G be a locally finite group of finite *c*-dimension k, and let S be the preimage in G of the socle of G/R, where R is the locally solvable radical of G. Is it true that the factor group G/S contains an abelian subgroup of index bounded by a function of k?

A. V. Vasil'ev

*Yes, it is true (A. A. Buturlakin, `arXiv:1805.00910`).

NEW ***19.109.** A subgroup H of a finite group G is called pronormal if for any $g \in G$ the subgroups H and H^g are conjugate by an element of $\langle H, H^g \rangle$. A maximal subgroup of a maximal subgroup is called second maximal. Is it true that in a non-abelian finite simple group G all maximal subgroups are Hall subgroups if and only if every second maximal subgroup of G is pronormal in G?

V. I. Zenkov

*No, not always, for example, in $SL_2(2^{11})$ every second maximal subgroup is pronormal (V. N. Tyutyanov, *Letter of 23 November 2018*).